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Exact Phase-Family Test of FRC Duality v0.1: A Lift Obstruction on the von Mises Manifold

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Abstract

> *The fixed-mean von Mises family provides an exact test of whether the FRC ledger involutions classified in FRC 830.001 lift from derived coordinates to probability distributions. The family has genuine information-geometric structure: concentration kappa and mean resultant r are natural and expectation coordinates, the log-partition function is strictly convex, and minus the differential entropy is its Legendre dual potential. That structure is not the FRC ledger exchange. For the ledger ell(kappa)=(S(kappa),ln r(kappa)), introduce u=S+ln r and v=S-ln r. The function v is strictly decreasing, while u is strictly unimodal with its unique maximum at kappa r=1. Because*

> *D(x,y)=(-y,-x) preserves v, every proposed D lift must fix kappa and can exist only at the two isolated roots of u=0. The broader half-plane involutions R_p(x,y)=(-x-2y+p,y) likewise force kappa to remain fixed and survive at no more than two isolated fixed ledgers. Neither class lifts on any open concentration domain. The point kappa r=1 is a maximum of the system-only ledger total, not a D-fixed point or structural self-duality. FRC therefore retains a P2 coordinate involution and gains a precise P3 lift obstruction in this family; it does not gain a phase-family physical or Majid-style self-duality.*

0. Result in one paragraph

The exact phase-family test returns an obstruction, not a self-duality. The von Mises family itself has substantial structure: its concentration κ and mean resultant r are natural and expectation coordinates of a strictly convex exponential family, related by a Legendre transform. But the FRC coordinate involution $D(S, \ln r) = (-\ln r, -S)$ is not that transform. A strictly decreasing function $v(\kappa) = S(\kappa) - \ln r(\kappa)$ is preserved by D, forcing any distribution-level lift to keep κ fixed. The commuting square then holds only at two isolated ledger-fixed concentrations. Every half-plane involution R_p fails for the same structural reason: it leaves $\ln r$ fixed, and r uniquely determines κ. The known point $\kappa r = 1$ is instead the unique maximum of $u(\kappa) = S(\kappa) + \ln r(\kappa)$. Thus FRC wins a precise theorem about where its ledger symmetry stops. It does not acquire a nontrivial map on the underlying phase distributions.

1. Question, scope, and status

FRC 830.001 proved that

$$D(x,y) = (-y,-x), \qquad D^2 = I, \qquad D^*(dx+dy) = -(dx+dy), \tag{1}$$

is the unique nonidentity exact affine involution of the nonnegative-entropy ledger cone $M_+ = [0,\infty) \times (-\infty, 0]$ within its declared class. It also classified the exact affine involutions of the entropy-unrestricted half-plane $M = \mathbb{R} \times (-\infty, 0]$ as the identity and

$$R_p(x,y) = (-x-2y+p, y), \qquad p \in \mathbb{R}. \tag{2}$$

Those theorems act on ledger coordinates. FRC 830.002 made the next burden explicit: a strong register morphism must include a state or model map whose ledger square commutes. This paper tests that burden on the fixed-mean von Mises family used in FRC 566.030.

The question is exact:

> *Does either D or a fixed R_p induce a nonidentity map on von Mises probability densities over a nonempty open concentration domain?*

The answer is no. The proof is analytic and does not depend on numerical root finding. Numerical values below only locate the isolated fixed ledgers and check the closed-form calculations.

The results are candidates for exact mathematical evidence pending independent proof review. They neither test an environment nor promote the canonical FRC bookkeeping relation to a universal physical law.

2. The statistical family and the lift interface

2.1 Fixed-mean von Mises family

Fix the mean direction at zero, the angle coordinate $\theta \in [0, 2\pi)$, and the reference measure $d\theta$. The canonical k^* is the scale-invariant Boltzmann bridge, while k^*_μ is its numerical representation in register μ. Declare the natural-information register μ_{nat}, where $k^*_{\mu_{\text{nat}}} = 1$. For $\kappa > 0$, define

$$p_\kappa(\theta) = \frac{e^{\kappa \cos\theta}}{2\pi I_0(\kappa)}, \qquad \psi(\kappa) = \ln\big(2\pi I_0(\kappa)\big). \tag{3}$$

The natural sufficient statistic is $T(\theta) = \cos\theta$. Its expectation coordinate and the differential entropy are

$$r(\kappa) = \mathbb{E}_\kappa[\cos\theta] = \frac{I_1(\kappa)}{I_0(\kappa)}, \qquad S(\kappa) = \psi(\kappa) - \kappa r(\kappa). \tag{4}$$

The phase-family ledger map is

$$\ell : (0, \infty) \longrightarrow M, \qquad \ell(\kappa) = \big(S(\kappa), \ln r(\kappa)\big). \tag{5}$$

Because $0 < r(\kappa) < 1$, the second coordinate is always negative. The first coordinate is a differential entropy and eventually becomes negative, so the full curve belongs to M but not entirely to M_+. Define the adapted ledger coordinates along the curve by

$$u(\kappa) = S(\kappa) + \ln r(\kappa), \qquad v(\kappa) = S(\kappa) - \ln r(\kappa). \tag{6}$$

Here u is the system-only quantity Q of FRC 566.030 in the natural-information register μ_{nat}, and v is the transverse ledger coordinate used in FRC 830.001.

2.2 Definitions

Term	Definition	Evidential meaning
Natural coordinate	The exponential-family parameter κ multiplying $\cos\theta$.	A parameter of the density, not a ledger coordinate.
Expectation coordinate	$r = \mathbb{E}_\kappa[\cos\theta]$.	An independently defined moment of the density.
Legendre dual potential	$\psi^*(r) = \sup_{\kappa>0}\{\kappa r - \psi(\kappa)\}$.	Genuine information-geometric structure.
Ledger lift	A map on concentrations or densities that induces a declared ledger transformation through a commuting square.	Moves the underlying statistical state before comparing ledger coordinates.
Nontrivial lift	A ledger lift defined on more than isolated fixed states and not equal to the identity state map everywhere.	Candidate structural duality evidence.
Isolated fixed ledger	One concentration whose ledger is fixed by a coordinate map.	A fixed point, not by itself an exchange of structures.
Structural duality	A compositional map on declared underlying structures inducing the coordinate action.	Promotion level P3 in FRC 830.000.
Empirical realization	A structural map implemented with independent observables, boundary, units, and dynamics.	Promotion level P4; not tested here.

2.3 Lift criterion

Let A be a candidate ledger map and let $E \subset (0,\infty)$. A lift within the fixed-mean von Mises family is a predeclared map

$$\tau : E \longrightarrow (0,\infty), \qquad p_\kappa \longmapsto p_{\tau(\kappa)}, \tag{7}$$

such that

$$\ell \circ \tau = A \circ \ell \quad \text{on } E. \tag{8}$$

Equation (8) is the strong commuting-square condition of FRC 830.002. A map defined only after selecting desired ledger outputs is not an operational adapter.

If the mean direction μ is restored in $p_{\kappa,\mu}(\theta) \propto e^{\kappa\cos(\theta-\mu)}$, both S and r remain independent of μ. Rotations of μ are therefore real density symmetries but are invisible to this ledger. They cannot supply a ledger action that the ledger itself does not determine. The one-parameter test fixes $\mu = 0$ so that this blind fiber is not mistaken for a lift of D.

3. Exact geometry of the von Mises ledger curve

Theorem 1. Strict statistical geometry and adapted-coordinate monotonicity

On $\kappa > 0$:

1. r is strictly increasing from 0 to 1;
2. v is strictly decreasing;
3. u has exactly one stationary point, a strict global maximum at the unique

κ_* satisfying $\kappa_* r(\kappa_*) = 1$; and

1. κ and r are Legendre-dual coordinates with

$\psi^*(r) = -S(\kappa)$.

Proof. Differentiation under the integral of the regular exponential family gives

$$\psi'(\kappa) = r(\kappa), \qquad r'(\kappa) = \psi''(\kappa) = \mathrm{Var}_\kappa(\cos\theta) > 0. \tag{9}$$

The variance is strictly positive because $\cos\theta$ is not almost surely constant under a strictly positive circle density. Hence r is strictly increasing. The standard endpoint limits are

$$r(\kappa) \sim \frac{\kappa}{2} \quad (\kappa \downarrow 0), \qquad r(\kappa) = 1 - \frac{1}{2\kappa} + O(\kappa^{-2}) \quad (\kappa \to \infty), \tag{10}$$

so its range is $(0, 1)$.

Differentiating (4) and (6) yields

$$S'(\kappa) = -\kappa r'(\kappa), \qquad \frac{d}{d\kappa} \ln r(\kappa) = \frac{r'(\kappa)}{r(\kappa)}, \tag{11}$$

and therefore

$$\frac{dS}{d \ln r} = -\kappa r, \tag{12}$$

the exact identity reported in FRC 566.030. For the adapted coordinates,

$$u'(\kappa) = \frac{r'(\kappa)}{r(\kappa)} \big(1 - \kappa r(\kappa)\big), \qquad v'(\kappa) = -r'(\kappa) \left(\kappa + \frac{1}{r(\kappa)}\right) < 0. \tag{13}$$

The function $h(\kappa) = \kappa r(\kappa)$ obeys

$$h'(\kappa) = r(\kappa) + \kappa r'(\kappa) > 0, \qquad \lim_{\kappa \downarrow 0} h = 0, \qquad \lim_{\kappa \to \infty} h = \infty. \tag{14}$$

It crosses 1 exactly once. Equation (13) then proves that u increases before that crossing and decreases after it, while v is injective on the whole domain.

Finally, strict convexity of ψ makes $r = \psi'(\kappa)$ a valid dual coordinate. At the unique corresponding $\kappa(r)$,

$$\psi^*(r) = \kappa r - \psi(\kappa) = -S(\kappa). \tag{15}$$

This is the ordinary Legendre structure of an exponential family. It exchanges natural and expectation descriptions; it does not exchange S with $-\ln r$. $\square$

Corollary 1. Exact domain of the cone involution

Since $S'(\kappa) < 0$, S decreases from $\ln(2\pi)$ to $-\infty$ and has one zero κ_0. Thus

$$K_+ = \{\kappa > 0 : S(\kappa) \geq 0\} = (0, \kappa_0], \qquad \kappa_0 \approx 17.5949792765. \tag{16}$$

Only K_+ maps into M_+, the declared domain of D. If $\kappa > \kappa_0$, the proposed transformed log coherence $-S(\kappa)$ is positive and cannot equal $\ln r(\kappa')$ for any von Mises concentration

$\kappa' > 0$.

4. The D lift obstruction

In the adapted coordinates, (1) acts as

$$D(u, v) = (-u, v). \tag{17}$$

The transverse coordinate v is therefore the decisive invariant.

Theorem 2. Complete classification of von Mises lifts of D

Let $E \subset K_+$ and suppose a map $\tau : E \to (0, \infty)$ satisfies

$$\ell(\tau(\kappa)) = D\ell(\kappa) \qquad (\kappa \in E). \tag{18}$$

Then every $\kappa \in E$ obeys

$$\tau(\kappa) = \kappa, \qquad u(\kappa) = 0. \tag{19}$$

The equation $u = 0$ has exactly two solutions $\kappa_- < \kappa_* < \kappa_+$. Consequently D has no lift on any nonempty open concentration interval and no lift containing a non-fixed concentration. Its only restrictions within the family are the identity density map at the two isolated fixed ledgers.

Proof. Equation (18) and (17) imply

$$u(\tau(\kappa)) = -u(\kappa), \qquad v(\tau(\kappa)) = v(\kappa). \tag{20}$$

Theorem 1 proves that v is strictly decreasing and hence injective. The second equality forces $\tau(\kappa) = \kappa$. The first then gives $u(\kappa) = -u(\kappa)$, so $u(\kappa) = 0$.

It remains to classify the roots. Standard small- and large-κ asymptotics give

$$u(\kappa) = \ln(\pi\kappa) + O(\kappa^2) \quad (\kappa \downarrow 0), \qquad u(\kappa) = \frac{1}{2} \ln \frac{2\pi e}{\kappa} + O(\kappa^{-1}) \quad (\kappa \to \infty). \quad (2$$

Thus $u \to -\infty$ at both ends. The maximum is positive by an exact lower bound. At $\kappa = 1$,

$$u(1) = \ln\big(2\pi I_1(1)\big) - r(1) > \ln \pi - 1 > 0, \tag{22}$$

because $I_1(1) > 1/2, 0 < r(1) < 1$, and $\pi > e$. Since Theorem 1 proves strict increase before κ_* and strict decrease after it, the intermediate value theorem gives exactly one zero on each side of κ_*. Neither zero can be an interval. At either zero, (19) makes the induced density action $p_\kappa \mapsto p_\kappa$, the identity. □

4.1 Reproducible locations of the fixed ledgers

Direct evaluation brackets the two exact roots as

$$\begin{aligned} u(0.33146) < 0 < u(0.33147), &\qquad \kappa_- \approx 0.3314630060, \\ u(16.56440) > 0 > u(16.56450), &\qquad \kappa_+ \approx 16.5644135142. \end{aligned} \tag{23}$$

At each root, $S = -\ln r > 0$, so both are inside K_+. The existence and uniqueness proof does not use the decimal values.

4.2 Why isolated fixed ledgers are not a structural lift

Restricting a coordinate involution to its fixed set always makes its action look like the identity. Here the only induced state maps are literally $p_{\kappa_-} \mapsto p_{\kappa_-}$ and $p_{\kappa_+} \mapsto p_{\kappa_+}$. No entropy observable is exchanged with a coherence observable, no natural coordinate is exchanged with an expectation coordinate, and no nonidentity operation acts on a neighborhood. These points are valid fixed ledgers but fail the nontriviality part of the P3 promotion test.

5. The half-plane involutions R_p

The family R_p of (2) is the complete nonidentity exact affine-involution class on M from FRC 830.001. Unlike D, it is defined even after the differential entropy becomes negative.

Theorem 3. Complete classification of von Mises restrictions of R_p

Fix $p \in \mathbb{R}$. If $E \subset (0, \infty)$ and $\tau : E \to (0, \infty)$ satisfy

$$\ell(\tau(\kappa)) = R_p \ell(\kappa), \tag{24}$$

then

$$\tau(\kappa) = \kappa, \qquad p = 2u(\kappa) \tag{25}$$

at every point of E. Let $u_* = u(\kappa_*)$. The possible solution counts are

$$\#\{\kappa : p = 2u(\kappa)\} = \begin{cases} 0, & p/2 > u_*, \\ 1, & p/2 = u_*, \\ 2, & p/2 < u_*. \end{cases} \tag{26}$$

Therefore no fixed R_p lifts on a nonempty open concentration interval. All surviving restrictions induce the identity on concentrations and densities.

Proof. The second coordinate of $R_p(x, y)$ is y. Equation (24) gives

$$\ln r(\tau(\kappa)) = \ln r(\kappa). \tag{27}$$

Strict monotonicity of r forces $\tau(\kappa) = \kappa$. Equality of the first coordinates then reads

$$S(\kappa) = -S(\kappa) - 2\ln r(\kappa) + p,$$

which is exactly (25). The endpoint limits and strict unimodality of u from Theorems 1-2 give (26). Each solution is isolated except that the middle case has a single tangency; none contains an open interval. □

6. The stationary point is not a duality point

Theorem 4. Exact status of kappa r = 1

The condition

$$\kappa_* r(\kappa_*) = 1 \tag{28}$$

selects the unique global maximum of $u = S + \ln r$. It does not select a fixed point of D. Numerically,

$$\begin{aligned} \kappa_* &\approx 1.6082794717, & r_* &\approx 0.6217824810, \\ S_* &\approx 1.4026220318, & u_* &\approx 0.9274570753 > 0. \end{aligned} \tag{29}$$

The half-plane involution R_{p_*} with

$$p_* = 2u_* \approx 1.8549141507 \tag{30}$$

has a fixed ledger leaf tangent to the phase-family curve at κ_*. This does not create a nontrivial lift.

Proof. Equation (13) makes (28) equivalent to $u'(\kappa_*) = 0$, and Theorem 1 proves this point is the unique maximum. Equation (22) proves $u_* > 0$ without numerical input. The fixed set of D is $u = 0$, so κ_* is not D-fixed.

For R_{p_*}, Theorem 3 reduces the fixed equation to $p_* = 2u(\kappa)$. At the maximum it has exactly one solution and $u' = 0$, which is the stated tangency. The induced map remains $\tau(\kappa_*) = \kappa_*$ and acts as the identity on the density. Choosing p_* after inspecting the maximum is also outcome-dependent fitting; even if p_* had been predeclared, the isolated identity restriction would still fail the nontrivial lift criterion. ☐

The stationary point retains the exact meaning assigned in FRC 566.030: $d(S + \ln r) = 0$ along this system-only family. No bath exists in the family, so the condition does not establish zero entropy production, an equality of production and export, or an environmental balance.

Corollary 2. Unit-representation invariance of the obstruction

Let μ be any lawful entropy-unit representation of the same fixed-mean von Mises register used above. Then

$$\frac{S_\mu(\kappa)}{k^*_\mu} = S_{\mu_{\mathrm{nat}}}(\kappa), \qquad \ln r_\mu(\kappa) = \ln r_{\mu_{\mathrm{nat}}}(\kappa).$$

Therefore every lawful entropy-unit representation has the same normalized ledger curve, the same stationary concentration, the same two fixed-ledger roots, and the same open-domain lift obstruction proved in Theorems 1--4.

Proof. A lawful conversion multiplies the numerical entropy and bridge by the same positive factor and leaves the coherence observable r unchanged. The two normalized ledger coordinates are therefore identical pointwise in κ. Every derivative, fixed-set equation, monotonicity result, and lift commuting square used in Theorems 1--4 is unchanged. ☐

This corollary concerns unit-equivalent presentations of the same register. It does not extend the obstruction to a different coherence channel, statistical family, boundary, or non-unit adapter.

7. Exhaustion of the candidate structures

Candidate	Underlying action tested	Exact result	Duality status

Identity I	$p_\kappa \mapsto p_\kappa$	Global lift.	Trivial identity, not a nonidentity duality.
D on M_+	A concentration map satisfying (18)	Only κ_- and κ_+; state action is identity.	P2 ledger involution; P3 lift rejected on this family.
R_p on M	A concentration map satisfying (24) for fixed p	Zero, one, or two isolated identity restrictions.	No open-domain P3 lift.
Positive multiples aI, aD	Cone automorphisms from 830.001	Not exact involutions unless $a = 1$; outside this paper's exact-duality test.	Constant-conformal coordinate maps only.
Other ambient affine involutions	Maps not preserving M or M_+	Fail the declared operational domain before the family test.	Not admissible phase-register automorphisms.
Mean-direction rotations	$p_{\kappa,\mu} \mapsto p_{\kappa,\mu+\phi}$	Genuine density actions invisible to S and r.	Fiber symmetry, not induced by D or R_p.
$\kappa \leftrightarrow r$ with $\psi \leftrightarrow \psi^*$	Natural/expectation coordinate change and Legendre transform	Exact information-geometric dual description.	Genuine dual-coordinate structure, but not FRC ledger self-duality.
von Mises inclusion into all phase densities	$\kappa \mapsto p_\kappa$ from 830.002	Exact strong embedding, not onto.	Valid nonidentity morphism, not an involutive duality.

This table exhausts the exact affine involutions passed by 830.001 after their domain restrictions. It does not claim a classification of every nonlinear map on probability distributions.

8. Relation to prior work

8.1 Von Mises, Kuramoto, and FRC 566.030

The mean resultant $r = I_1/I_0$ is standard circular-statistics mathematics and the continuum form of the Kuramoto order parameter. FRC 566.030 correctly identified the exact derivative relation (12) and the unique stationary point (28). The new result is not a new name for those facts. It is the lift test: the ledger involutions cannot induce a nonidentity state map on this family.

8.2 Information geometry

Information geometry treats a regular exponential family through natural and expectation coordinates, a strictly convex log-partition potential, and its Legendre dual. Equations (9) and (15) instantiate that standard structure exactly. The distinction matters: the von Mises family already possesses a real dual-coordinate geometry. The failed FRC construction is the different claim that $(S, \ln r)$ is self-dual under a signed exchange.

8.3 Bessel functions

The series, derivative identities, and small/large argument asymptotics of the modified Bessel functions are standard. The proof uses them only for endpoint behavior and exact positivity; all lift conclusions then follow from monotonicity and injectivity.

8.4 Majid-style self-duality

Majid's representation-theoretic self-duality names the algebraic or representational structures being paired and supplies maps acting at that level. Here the genuine structure is an exponential-family manifold with Legendre-dual coordinates. The FRC map D acts only on two derived scalar channels and fails to move even the natural parameter. The result therefore supports Majid's methodological standard—lift the map to the structures—but does not realize a Hopf-algebraic, representation-theoretic, or quantum Born self-duality.

9. Claim-status table and kill result

ID	Statement	Status	Basis
D1	Equations (3)-(8) define the fixed-mean phase family, ledger, and lift criterion.	Definition	Declared statistical register.
U1	The normalized von Mises ledger and its lift obstruction are invariant under lawful entropy-unit representation changes.	Proved corollary	Entropy and k_μ^* transform together while r is unchanged.
T1	r increases, v decreases, u is strictly unimodal, and $\psi^* = -S$.	Proved theorem	Variance identity and exact differentiation.
C1	The D cone domain is $K_+ = (0, \kappa_0]$.	Proved corollary	Strict decrease and endpoint limits of S.
T2	Every D lift fixes κ and exists only at the two roots of $u = 0$.	Proved obstruction	Injectivity of v and strict unimodality of u.

		theorem	
N1	$\kappa_- \approx 0.3314630060$ and $\kappa_+ \approx 16.5644135142$.	Reproducible numerical location	Bessel series and deterministic bisection; not used for proof.
T3	Every fixed R_p restriction fixes κ and has zero, one, or two isolated solutions.	Proved obstruction theorem	Injectivity of r and level sets of u.
T4	$\kappa r = 1$ is the unique maximum of u, not a D-fixed point.	Proved theorem	Equations (13), (22), and (28).
G1	κ and r are Legendre-dual coordinates.	Established standard structure	Strict convexity and (15).
R1	The isolated identity restrictions are nontrivial FRC dualities.	Rejected	They move neither concentration nor density.
R2	D or R_p supplies a P3 structural lift on an open von Mises domain.	Rejected for this family	Theorems 2-3.
O1	A different FRC register possesses a nontrivial structural duality.	Open	Requires its own independently defined state-space map.
O2	A driven open system produces a connection or holonomy beyond ledger bookkeeping.	Open	Assigned to FRC 830.004.

The paper's kill condition is met on the obstruction branch. The surviving affine involutions do not lift from $(S, \ln r)$ to a nonidentity action on the von Mises distributions or either of their natural/expectation coordinates. Therefore they are not promoted from P2 ledger geometry to P3 structural duality in this arena.

The obstruction does not erase valid results. FRC retains:

- the exact affine classification and D involution at P2;
- the exact von Mises/Kuramoto formulas;
- the strong inclusion of the von Mises family into the phase-distribution

register; and

- the family's genuine Legendre-dual statistical coordinates.

What fails is the extra claim that the FRC ledger exchange is itself the family's structural self-duality.

10. What this paper does not establish

This paper does not establish:

- a bath, entropy-production rate, or entropy-export channel;
- a universal physical law for $dS + k^* d \ln C = 0$;
- a nontrivial FRC duality on the von Mises family;
- that every nonlinear or cross-family probability map is impossible;
- a full classification of the two-parameter (κ, μ) family;
- that mean-direction rotations realize the ledger involution;
- a quantum, Hopf-algebraic, representation-theoretic, Born, or Majid-style

self-duality;

- a result for purity, interference, transport, or another coherence route;
- an environmental interpretation of $\kappa r = 1$; or
- independent mathematical review.

FRC 830.004 must now take a different route. It should begin from an explicit driven open-system model, independently define production and exchange channels, and test whether a boundary-relative form has path dependence, curvature, or holonomy. It may not reuse the failed phase-family ledger lift as evidence for those structures.

Appendix A. Exact and numerical reproducibility

The proof uses no fitted parameter and no simulation. The modified Bessel series used for numerical checking is

$$I_n(\kappa) = \sum_{m=0}^{\infty} \frac{(\kappa/2)^{2m+n}}{m!\,\Gamma(m+n+1)}. \tag{31}$$

For each displayed numerical value:

1. evaluate I_0 and I_1 by (31) in IEEE-754 binary64 arithmetic;
2. stop when the next positive term is at most 10^{-16} times the updated

partial sum;

1. compute r, S, u, v, and $\kappa r - 1$ from (4) and (6); and

2. bisect each declared bracket until its width is below 10^{-12}.

The decisive sign checks are

$$\begin{array}{c|r|r} \kappa & u(\kappa) & \kappa r(\kappa)-1 \\ \hline 0.33146 & -8.3472\times10^{-6} & -9.4581\times10^{-1} \\ 0.33147 & +1.9421\times10^{-5} & -9.4580\times10^{-1} \\ 1.60827 & +9.2746\times10^{-1} & -9.3438\times10^{-6} \\ 1.60829 & +9.2746\times10^{-1} & +1.0386\times10^{-5} \\ 16.56440 & +3.9527\times10^{-7} & +1.5056\times10^{1} \\ 16.56450 & -2.5296\times10^{-6} & +1.5056\times10^{1} \end{array} \tag{32}$$

The analytic proof of Theorems 1-4 is independent of the stopping tolerance, root solver, and rounded values.

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